The NN System in Three Dimensions

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General Form of NN interaction

- **Space (e.g. momenta)**
  - Basis: vector variables $\vec{p}\,'-\vec{p},\quad \vec{p}\,'+\vec{p},\quad \vec{p}\,'\times\vec{p}$
- **Spin – Operators** $\vec{\sigma}_1$ and $\vec{\sigma}_2$
- **Isospin - Operators** $\vec{\tau}_1$ and $\vec{\tau}_2$

Idea: form scalar functions with the vector variables

handle operators analytically

General Form of NN interaction, cont'd

Allow explicit isospin dependence: $\langle t'm_t' \mid V \mid tm_t \rangle = \delta_{tt'}\delta_{m_t m_t'} V^{tm_t}$

Spin momentum operator structure invariant under rotation, parity, time-reversal

$$
\begin{aligned}
w_1(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= 1 \\
w_2(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \\
w_3(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= i\,(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathrm{p} \times \mathrm{p}') \\
w_4(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p} \times \mathrm{p}')\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p} \times \mathrm{p}') \\
w_5(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p}' + \mathrm{p})\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p}' + \mathrm{p}) \\
w_6(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p}' - \mathrm{p})\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p}' - \mathrm{p})
\end{aligned}
$$

Most general expression for any NN potential:

$$
\boxed{V^{tm_t} \equiv \sum_{j=1}^{6} v_j^{tm_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})}
$$

where $v_j^{tm_t}(\mathbf{p}', \mathbf{p})$ is a scalar function of |p'|, |p|, and p'·p

Examples:

χEFT LO potential:

$$V_{LO} = -\frac{1}{(2\pi)^3}\frac{g_A^2}{4F_\pi^2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{q}\,\boldsymbol{\sigma}_2\cdot\mathbf{q}}{\mathbf{q}^2+M_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \frac{C_S}{(2\pi)^3} + \frac{C_T}{(2\pi)^3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,,$$

$$\mathbf{q} = \mathbf{p}\,' - \mathbf{p}$$

χEFT NLO potential:

$$\begin{aligned} V_{NLO} \;=\; & -\frac{1}{(2\pi)^3}\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{384\pi^2 F_\pi^4}\,L^{\tilde\Lambda}(q)\left[4m_\pi^2(5g_A^4-4g_A^2-1)+\mathbf{q}^{\,2}(23g_A^4-10g_A^2-1)+\frac{48g_A^4m_\pi^4}{4m_\pi^2+\mathbf{q}^{\,2}}\right] \\ & -\frac{1}{(2\pi)^3}\frac{3g_A^4}{64\pi^2F_\pi^4}\,L^{\tilde\Lambda}(q)\left(\boldsymbol{\sigma}_1\cdot\mathbf{q}\,\boldsymbol{\sigma}_2\cdot\mathbf{q}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,\mathbf{q}^{\,2}\right) \\ & +\frac{C_1}{(2\pi)^3}\,\mathbf{q}^{\,2}+\frac{C_2}{(2\pi)^3}\,\mathbf{k}^2+\left(\frac{C_3}{(2\pi)^3}\,\mathbf{q}^{\,2}+\frac{C_4}{(2\pi)^3}\,\mathbf{k}^2\right)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ & +\frac{C_5}{(2\pi)^3}\,\frac{i}{2}\,(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{q}\times\mathbf{k}+\frac{C_6}{(2\pi)^3}\,\mathbf{q}\cdot\boldsymbol{\sigma}_1\,\mathbf{q}\cdot\boldsymbol{\sigma}_2+\frac{C_7}{(2\pi)^3}\,\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2\,, \end{aligned} \tag{C2}$$

$$\mathbf{k} \;=\; \tfrac{1}{2}\,(\mathbf{p}\,'+\mathbf{p}\,)$$

NN t-matrix:

$$t^{tm_t} = V^{tm_t} + V^{tm_t} G_0 t^{tm_t}$$

$$t^{tm_t} \equiv \sum_{j=1}^{6} t_j^{tm_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})$$

$$V^{tm_t} \equiv \sum_{j=1}^{6} v_j^{tm_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})$$

$$\begin{aligned}\sum_{j=1}^{6} t_j^{tm_t}(\mathbf{p}', \mathbf{p})\,\mathrm{w}_j(\mathbf{p}', \mathbf{p}) \quad &= \quad \sum_{j=1}^{6} v_j^{tm_t}(\mathbf{p}', \mathbf{p})\mathrm{w}_j(\mathbf{p}', \mathbf{p}) \\ &+2\mu \lim_{\epsilon\to 0} \sum_{k,j=1}^{6} \int d\mathbf{p}'' \frac{v_k^{tm_t}(\mathbf{p}', \mathbf{p}'')\mathrm{w}_k(\mathbf{p}', \mathbf{p}'') t_j^{tm_t}(\mathbf{p}'', \mathbf{p})\,\mathrm{w}_j(\mathbf{p}'', \mathbf{p})}{p^2 + i\epsilon - p''^2}\end{aligned}$$

Project with w_k from the left
and perform the trace in NN spin space

$$\sum_j A_{kj}(\mathbf{p}',\mathbf{p})t_j^{tm_t}(\mathbf{p}',\mathbf{p}) = \sum_j A_{kj}(\mathbf{p}',\mathbf{p})v_j^{tm_t}(\mathbf{p}',\mathbf{p})$$
$$+ \int d^3p'' \sum_{jj'} v_j^{tm_t}(\mathbf{p}',\mathbf{p}'')G_0(p'')\; t_{j'}^{tm_t}(\mathbf{p}'',\mathbf{p})\; B_{kjj'}(\mathbf{p}',\mathbf{p}'',\mathbf{p})$$

All functions are scalar !

$$A_{kj}(\mathbf{p}',\mathbf{p}) \;\equiv\; \mathrm{Tr}\Big(w_k(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})\; w_j(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})\Big)$$

$$B_{kjj'}(\mathbf{p}',\mathbf{p}'',\mathbf{p}) \;\equiv\; \mathrm{Tr}\Big(w_k(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})\; w_j(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p}'')\; w_{j'}(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}'',\mathbf{p})\Big)$$

NN t-matrix consists of 6 coupled eqs of scalar functions

Structure of some of the A_{kj} and $B_{kjj'}$:

$$
\begin{aligned}
A_{26}(\mathbf{p}', \mathbf{p}) &= 4(\mathbf{p}' - \mathbf{p})^2 \\
A_{33}(\mathbf{p}', \mathbf{p}) &= -8(\mathbf{p} \times \mathbf{p}')^2 \\
A_{56}(\mathbf{p}', \mathbf{p}) &= 4(p'^2 - p^2)^2
\end{aligned}
$$

14 non-vanishing

$$
\begin{aligned}
B_{261}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= 4(\mathbf{p}' - \mathbf{p}'')^2 \\
B_{612}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= 4(\mathbf{p}' - \mathbf{p})^2 \\
B_{133}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= -8(\mathbf{p}'' \times \mathbf{p}') \cdot (\mathbf{p} \times \mathbf{p}'') \\
B_{331}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= -8(\mathbf{p} \times \mathbf{p}') \cdot (\mathbf{p}'' \times \mathbf{p}') \\
B_{313}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= -8(\mathbf{p} \times \mathbf{p}') \cdot (\mathbf{p} \times \mathbf{p}'') \\
B_{145}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= 4\{(\mathbf{p} \times \mathbf{p}') \cdot \mathbf{p}''\}^2 \\
B_{155}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= 4\{(\mathbf{p}' + \mathbf{p}'') \cdot (\mathbf{p}'' + \mathbf{p})\}^2 \\
B_{551}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) &= 4\{(\mathbf{p}' + \mathbf{p}) \cdot (\mathbf{p}' + \mathbf{p}'')\}^2
\end{aligned}
$$

148 non-vanishing

General form of scattering amplitude (Wolfenstein representation)

$$M = a + c(\sigma^{(1)} + \sigma^{(2)})\hat{N} + m(\sigma^{(1)}\hat{N})(\sigma^{(2)}\hat{N}) + (g+h)(\sigma^{(1)}\hat{P})(\sigma^{(2)}\hat{P}) + (g-h)(\sigma^{(1)}\hat{K})(\sigma^{(2)}\hat{K}) .$$

$$\hat{K} \equiv (q' - q)/|q' - q|$$
$$\hat{P} \equiv (q + q')/|q + q'|$$
$$\hat{N} \equiv (q \times q')/|q \times q'| .$$

Solution of our T-matrix equation can be directly mapped to the Wolfenstein amplitudes (initial state needs to be antisymmetrized)

Wolfenstein amplitudes lead directly to observables

(see e.g. N. Hoshizaki, Prog. Theor. Phys. 42, 107 (1968))

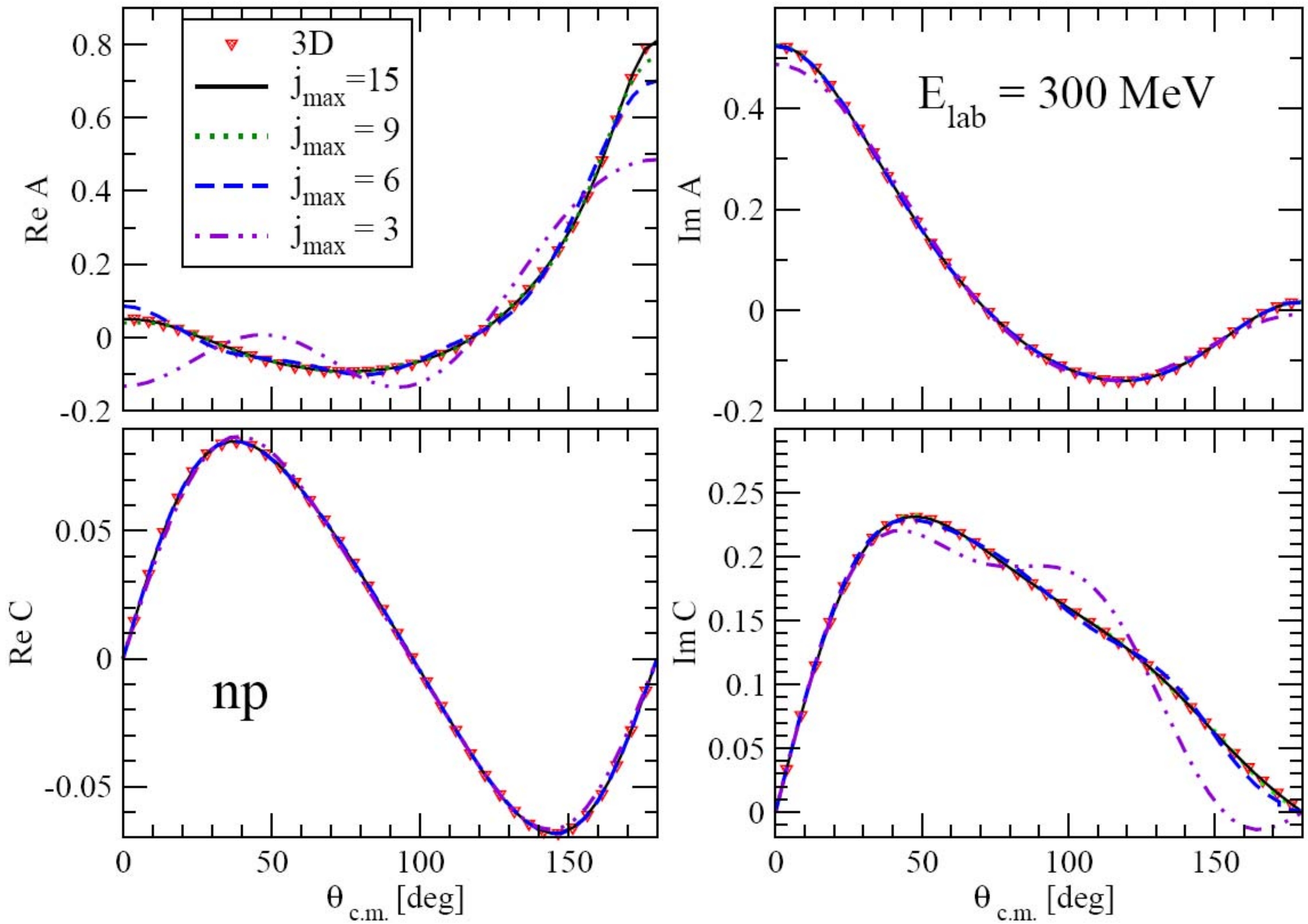
Wolfenstein Amplitudes:

$$
\begin{aligned}
a^{tm_t} &= \frac{1}{4}\,\mathrm{Tr}\,(M) \\
c^{tm_t} &= -i\frac{1}{8}\,\mathrm{Tr}\left(M\,\frac{w_3(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})}{|\mathbf{p}\times\mathbf{p}'|}\right) \\
m^{tm_t} &= \frac{1}{4}\,\mathrm{Tr}\left(M\,\frac{w_4(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})}{|\mathbf{p}\times\mathbf{p}'|^2}\right) \\
(g+h)^{tm_t} &= \frac{1}{4}\,\mathrm{Tr}\left(M\,\frac{w_5(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})}{(\mathbf{p}+\mathbf{p}')^2}\right) \\
(g-h)^{tm_t} &= \frac{1}{4}\,\mathrm{Tr}\left(M\,\frac{w_6(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})}{(\mathbf{p}-\mathbf{p}')^2}\right)
\end{aligned}
$$

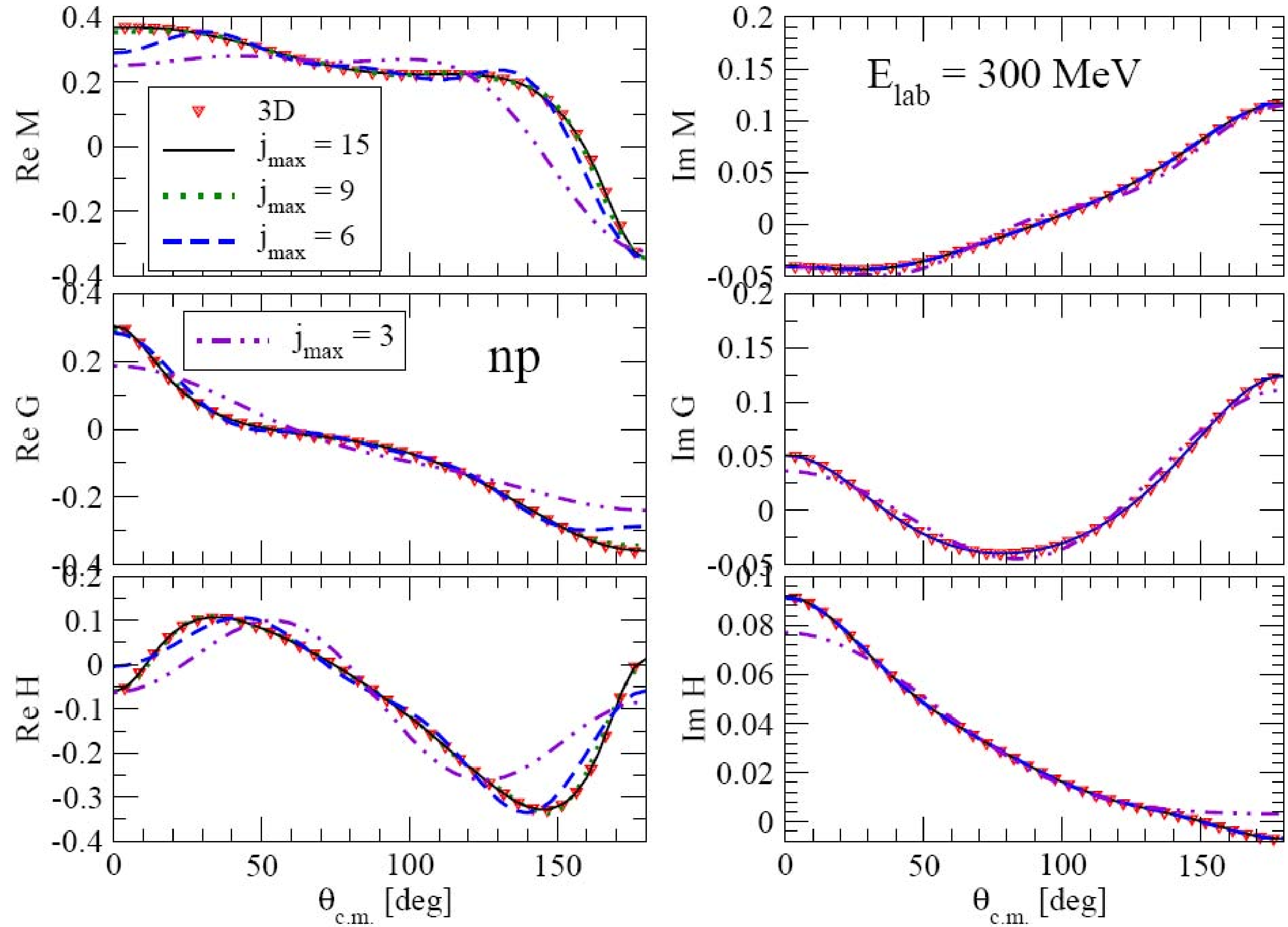
with

$$
\begin{aligned}
M^{tm_t}_{m_1'm_2',m_1m_2} &= -\frac{m}{2}(2\pi)^2\sum_{j=1}^{6}\Big[t_j^{tm_t}(\mathbf{p}',\mathbf{p})\,\langle m_1'm_2'|w_j(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',\mathbf{p})|m_1m_2\rangle \\
&+\,(-)^t t_j^{tm_t}(\mathbf{p}',-\mathbf{p})\,\langle m_1'm_2'|w_j(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathbf{p}',-\mathbf{p})|m_2m_1\rangle\Big]
\end{aligned}
$$

np scattering at 300 MeV – Bonn B

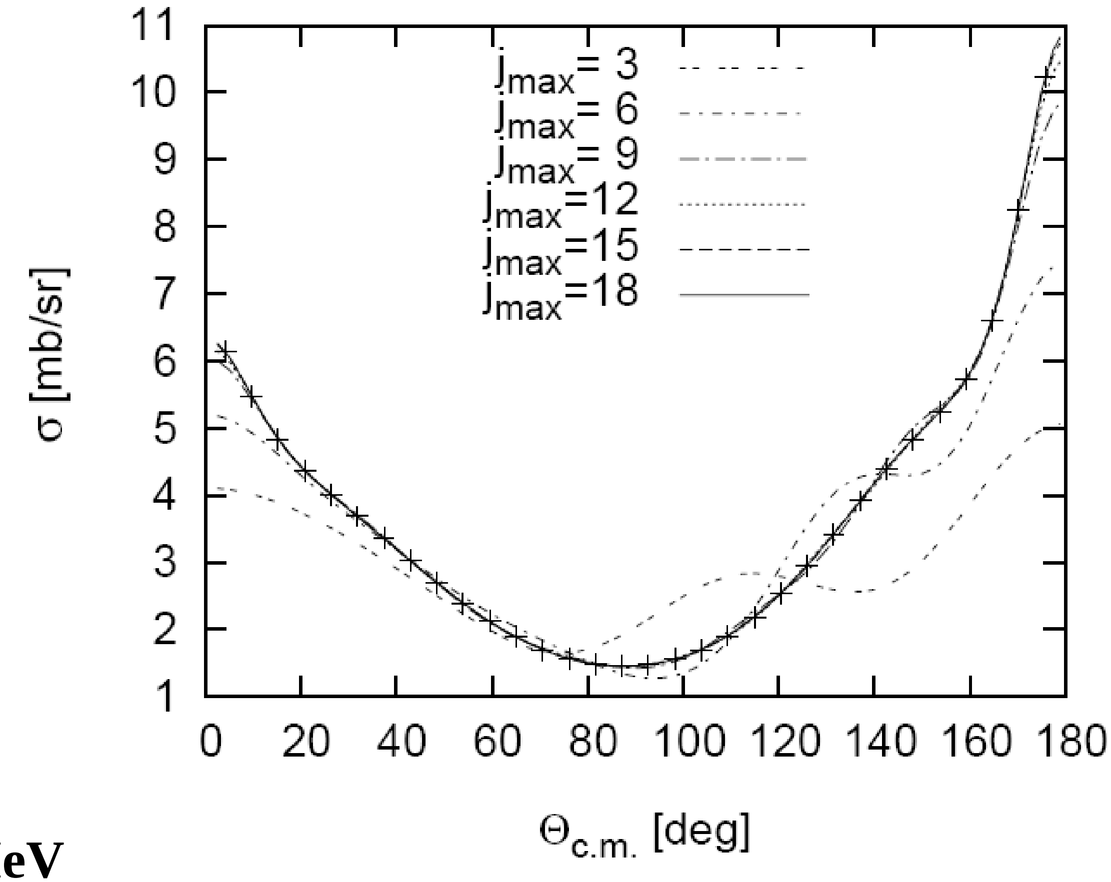


np scattering at 300 MeV – Bonn B



Spin averaged differential cross section:

$$I_0 = \frac{1}{4}$$



E_{lab} = 300 MeV

Bonn-B

Deuteron: t=0, s=1

$$\langle \mathrm{p}|\Psi_{m_d}\rangle = \left[\phi_1(p) + \left(\boldsymbol{\sigma}_1\cdot\mathrm{p}\,\boldsymbol{\sigma}_2\cdot\mathrm{p} - \frac{1}{3}p^2\right)\phi_2(p)\right]|1m_d\rangle$$
$$\equiv \sum_{k=1}^{2}\phi_k(p)\, b_k(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathrm{p})|1m_d\rangle,$$

Standard s-wave $\psi_0(p) = \phi_1(p),$

d-wave $\psi_2(p) = \frac{4p^2}{3\sqrt{2}}\phi_2(p).$

Schrödinger equation: $\Psi_{m_d} = G_0 V^{00}\Psi_{m_d}.$

$$\left[\phi_1(p) + \left(\boldsymbol{\sigma}_1\cdot\mathrm{p}\,\boldsymbol{\sigma}_2\cdot\mathrm{p} - \frac{1}{3}p^2\right)\phi_2(p)\right]|1m_d\rangle =$$
$$\frac{1}{E_d - \frac{p^2}{m}}\int d^3p' \sum_{j=1}^{6} v_j^{00}(\mathrm{p},\mathrm{p}')\, w_j(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\mathrm{p},\mathrm{p}')$$
$$\times\left[\phi_1(p') + \left(\boldsymbol{\sigma}_1\cdot\mathrm{p}'\,\boldsymbol{\sigma}_2\cdot\mathrm{p}' - \frac{1}{3}p'^2\right)\phi_2(p')\right]|1m_d\rangle,$$

Project from left with $\langle 1m_d | b_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p})$ **and sum over** $\mathbf{m_d}$

$$\sum_{m_d=-1}^{1} \langle 1m_d | b_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}) \sum_{k'=1}^{2} \phi_{k'}(p) b_{k'}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}) | 1m_d \rangle =$$

$$\frac{1}{E_d - \frac{p^2}{m}} \sum_{m_d=-1}^{1} \int d^3p' \sum_{j=1}^{6} v_j^{00}(\mathrm{p}, \mathrm{p}') w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}, \mathrm{p}') \sum_{k''=1}^{2} \phi_{k''}(p')\, b_{k''}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}') | 1m_d \rangle$$

With scalar functions:

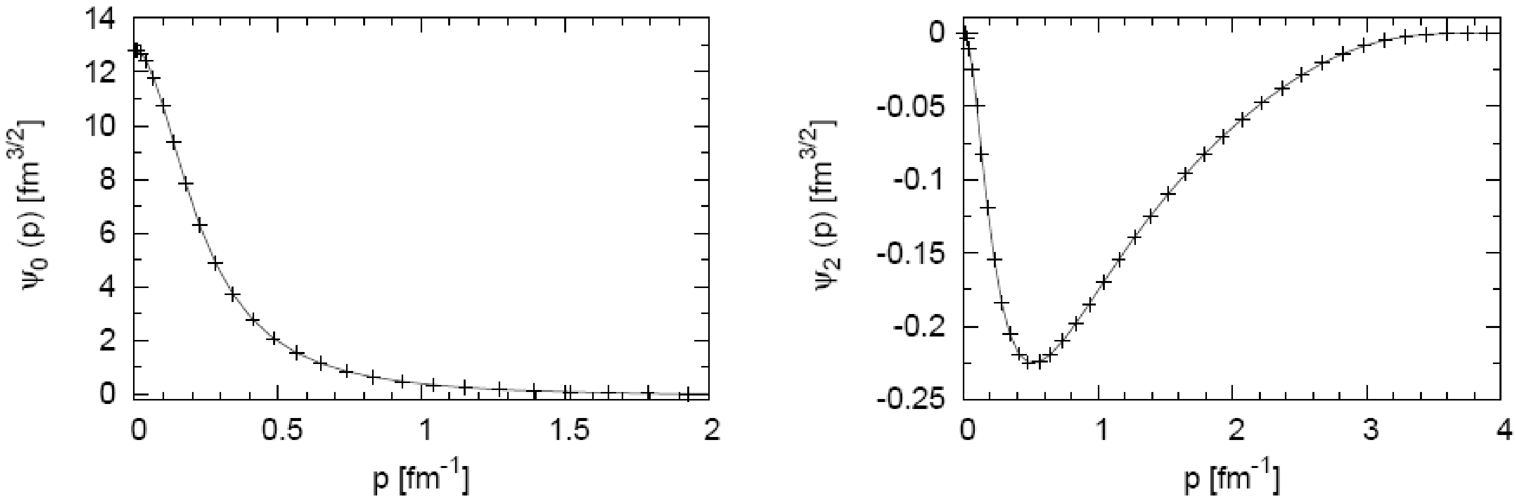
$$A^d_{kk'}(p) \equiv \sum^{1} \langle 1m_d | b_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}) b_{k'}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}) | 1m_d \rangle$$

$$B^d_{kjk''}(\mathrm{p}, \mathrm{p}') \equiv \sum_{m_d=-1}^{1} \langle 1m_d | b_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}) w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}, \mathrm{p}') b_{k''}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}') | 1m_d \rangle$$

Deuteron: coupled eq. of 2 scalar functions:

$$\sum_{k'=1}^{2} A^d_{kk'}(p) \phi_{k'}(p) = \frac{1}{E_d - \frac{p^2}{m}} \int d^3p' \sum_{j=1}^{6} v_j^{00}(\mathrm{p}, \mathrm{p}') \sum_{k''=1}^{2} B^d_{kjk''}(\mathrm{p}, \mathrm{p}') \phi_{k''}(p')$$

Example: S- and D-wave of chiral NNLO potential.



Published in **Phys.Rev.C81, 034006, 2010**

The Two-Nucleon System in Three Dimensions.

J. Golak, (Jagiellonian U.) , W. Glockle, (Ruhr U., Bochum) , R. Skibinski, H. Witala, D. Rozpedzik, K. Topolnicki, (Jagiellonian U.) , I. Fachruddin, (Indonesia U.) , Ch. Elster, (Ohio U.) , A. Nogga, (Julich, Forschungszentrum)

Summary

- Realize the most general operator form of
  - NN potential
  - deuteron wave function
  - NN t-matrix
- and derive coupled equations of scalar functions (of vector variables)
  - Deuteron: 2 equations
  - NN t-matrix: 6 equations
- Spin-momentum operators are algebraically treated by taking traces

Future: ^{3}H and ^{3}He :

$$\psi_{tT}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{8} \phi_{tT}^{(i)}(\boldsymbol{p},\boldsymbol{q})\; O_i|\chi^m\rangle$$

$$|\chi^m\rangle \;=\; |(0\tfrac{1}{2})\tfrac{1}{2}m\rangle$$

Fachruddin, Glöckle, Elster, Nogga PRC 69, 064002 (2004)

$$\begin{aligned}
O_1 &= 1 \\
O_2 &= \boldsymbol{\sigma}(23)\cdot\boldsymbol{\sigma}_{(1)} \\
O_3 &= \boldsymbol{\sigma}_{(1)}\cdot(\hat{p}\times\hat{q}) \\
O_4 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\times\hat{q} \\
O_5 &= \boldsymbol{\sigma}(23)\cdot\hat{q}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{p} \\
O_6 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{q} \\
O_7 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{p} \\
O_8 &= \boldsymbol{\sigma}(23)\cdot\hat{q}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{q}.
\end{aligned}$$

Apply this to Faddeev equations outlined in

Glöckle, Elster, Golak, Skibinski Witala, Kamada – FBS 47, 25 (2010)

Isospin states : $\left|\left(0\tfrac{1}{2}\right)\tfrac{1}{2}\right\rangle, \left|\left(1\tfrac{1}{2}\right)\tfrac{1}{2}\right\rangle, \left|\left(1\tfrac{1}{2}\right)\tfrac{3}{2}\right\rangle$

$$\boldsymbol{\sigma}(23) \equiv \tfrac{1}{2}(\boldsymbol{\sigma}_{(2)} - \boldsymbol{\sigma}_{(3)})$$